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Uniqueness of the Ground State in Weak Perturbations of Non-Interacting Gapped Quantum Lattice Systems

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We consider a general weak perturbation of a non-interacting quantum lattice system with a non-degenerate gapped ground state. We prove that in a finite volume the dependence of the ground state on the boundary condition exponentially decays with the distance to the boundary, which implies in particular that the infinite-volume ground state is unique. Also, equivalent forms of boundary conditions for ground states of general finite quantum systems are discussed.

KEY WORDS: Ground state; quantum lattice system.

1. INTRODUCTION AND RESULTS

We consider a quantum system on a lattice, which is a weak perturbation of a non-interacting system with a non-degenerate gapped ground state. A rigorous perturbation theory for such models, relying on a suitable ansatz for the ground state or zero-temperature path space expansions, was developed in refs. 2, 3, 5–7 and 10. In particular, it is known that the weakly interacting model has a ground state with a spectral gap, which can be obtained as the thermodynamic weak*-limit of ground states corresponding to finite volume restrictions of the Hamiltonian with empty or periodic boundary conditions. The question we address in the present paper is the uniqueness of the infinite volume ground state. In general, a ground state in a finite volume can be sensitive to boundary conditions, which can lead to different infinite volume states in the limit. We prove that for the model in question this is not the case.

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For translation-invariant ground states the uniqueness was proved by Matsui in refs. 8 and 9. His method relies on the specific energy functional, and the translational invariance is very essential for the proof. Intuitively one would expect, however, that the uniqueness should hold regardless of this invariance, as long as the perturbation is uniformly weak enough. We give a different proof, where the translational invariance plays no role. We consider ground states with most general boundary conditions in a finite volume and show that the dependence of such a state on the boundary condition exponentially decays with the distance to the boundary. The infinite volume uniqueness then follows as a straightforward consequence. Physically this result is quite natural and expected, but the mathematically rigorous analysis of ground states with general quantum boundary conditions is not obvious. Though in this article we restrict our attention to a model with a single non-degenerate ground state, we expect that the method developed here can be used to study completeness in more general situations, where, in particular, ground states with spontaneously broken translation symmetry (domain walls) can be present.

We give now precise statements.

We consider a quantum "spin" system on the lattice $\mathbb{Z}^\nu$. Suppose that for each $x \in \mathbb{Z}^\nu$ there is a Hilbert space $\mathcal{H}_x$ (possibly infinite-dimensional) associated with this site. For the restriction to a finite volume $\Lambda \subset \mathbb{Z}^\nu$ we will use the notation

$$\mathcal{H}_\Lambda := \otimes_{x \in \Lambda} \mathcal{H}_x.$$

The (formal) Hamiltonian of the model consists of a trivial non-interacting Hamiltonian and a perturbation

$$H = H_0 + \Phi.$$

Here H_0 is the non-interacting Hamiltonian

$$H_0 = \sum_{x \in \mathbb{Z}^\nu} h_x.$$

We assume that each h_x is a non-negative self-adjoint, possibly unbounded operator on $\mathcal{H}_x$ with a non-degenerate ground state $\Omega_x \in \mathcal{H}_x$

$$h_x \Omega_x = 0$$

and a uniform in x spectral gap

$$h_x|_{\mathcal{H}_x \ominus \Omega_x} \geqslant \mathbf{1} \tag{1}$$

(this is necessary and sufficient in order that the non-interacting Hamiltonian have a non-degenerate ground state and a spectral gap $\geqslant 1$; here and in the sequel we slightly abuse the notation by denoting the one-dimensional subspace spanned by Ω_x with the same symbol). In order to define the perturbation Φ we fix a finite subset $\Lambda_0 \subset \mathbb{Z}^\nu$ (range of the perturbation) and set

$$\Phi = \sum_{x \in \mathbb{Z}^\nu} \phi_x, \tag{2}$$

where ϕ_x is a self-adjoint bounded operator on $\mathcal{H}_{\Lambda_0+x}$ (here $\Lambda_0 + x$ is a shift of Λ_0). We will assume that the perturbation is small in the sense that $\sup_{x\in\mathbb{Z}^\nu} \|\phi_x\|$ is finite and small enough.

The existence of a ground state for such systems can be handled as follows. Let $\Lambda \subset \mathbb{Z}^\nu$ be a finite volume and H_Λ the restriction of the Hamiltonian H to Λ with empty boundary conditions

$$H_\Lambda := H_{\Lambda,0} + \Phi_\Lambda,$$

where

$$H_{\Lambda,0} := \sum_{x\in\Lambda} h_x, \quad \Phi_\Lambda := \sum_{x\in\mathbb{Z}^\nu : \Lambda_0 + x \subset \Lambda} \phi_x. \tag{3}$$

Since Φ_Λ is bounded, H_Λ is self-adjoint with $\mathrm{Dom}\,(H_\Lambda) = \mathrm{Dom}\,(H_{\Lambda,0})$. In what follows we use the local algebra:

$$\mathcal{A}_\infty := \bigcup_{\Lambda\subset\mathbb{Z}^\nu, |\Lambda|<\infty} \mathcal{B}(\mathcal{H}_\Lambda),$$

where for any finite Λ by $\mathcal{B}(\mathcal{H}_\Lambda)$ we denote the algebra of bounded operators in $\mathcal{H}_\Lambda$ ($\mathcal{A}$ is a C^*-inductive limit of $\mathcal{B}(\mathcal{H}_\Lambda)$; see ref. 2 for generalities on quasi-local C^*-algebras).

The following theorem was proved in ref. 10 (also see refs.11 and 6):

Theorem 1. There exists a constant c, depending only on the perturbation range Λ_0, such that if $\sup_x \|\phi_x\| < c$ then for any finite Λ the Hamiltonian H_Λ has a non-degenerate ground state $\Omega_\Lambda \in \mathcal{H}_\Lambda$ with a spectral gap $\geqslant 1/2$

$$H_\Lambda \Omega_\Lambda = E_\Lambda \Omega_\Lambda, \quad H_\Lambda|_{\mathcal{H}_\Lambda \ominus \Omega_\Lambda} \geqslant (E_\Lambda + 1/2)\mathbf{1}.$$

Moreover, let $\omega_{gs,\Lambda}$ be the corresponding ground state on $\mathcal{B}(\mathcal{H}_\Lambda)$

$$\omega_{gs,\Lambda}(A) := (A\Omega_\Lambda, \Omega_\Lambda)$$

(Ω_Λ is assumed to be normalized). Let $\Lambda \nearrow \mathbb{Z}^\nu$ mean that Λ converges to $\mathbb{Z}^\nu$ in the sense that it eventually contains any finite subset. Then there exists a state $\omega_{gs,\infty}$ on the local algebra $\mathcal{A}_\infty$, which is the thermodynamic weak*-limit of the finite-volume ground states

$$\omega_{gs,\Lambda}(A) \xrightarrow{\Lambda \nearrow \mathbb{Z}^\nu} \omega_{gs,\infty}(A) \quad \text{for any } A \in \mathcal{A}_\infty.$$

In order to discuss possible non-uniqueness of the ground state we need a general definition of a ground state. We define it using the standard local stability condition (see ref. 2).

Recall first that if G is a (possibly unbounded) self-adjoint and A a bounded operators acting in the same Hilbert space then, by definition, their commutator $[G, A]$ is a bounded operator B iff $\mathrm{Dom}\,(G)$ is invariant under A and

$$GAv - AGv = Bv$$

for all $v \in \mathrm{Dom}\,(G)$. For any finite Λ let

$$\mathcal{D}_\Lambda := \{A \in \mathcal{B}(\mathcal{H}_\Lambda) : [H_{\Lambda,0}, A] \text{ is bounded}\},$$

where $H_{\Lambda,0}$ is defined in (3). Clearly, $\mathcal{D}_{\Lambda_1} \subset \mathcal{D}_{\Lambda_2}$ for $\Lambda_1 \subset \Lambda_2$, so we can define

$$\mathcal{D} := \bigcup_{\Lambda \subset \mathbb{Z}^\nu, |\Lambda| < \infty} \mathcal{D}_\Lambda \subset \mathcal{A}_\infty.$$

For any Λ consider the set $\overline{\Lambda}$ obtained by adding all sites interacting with Λ

$$\overline{\Lambda} := \bigcup_{x \in \mathbb{Z}^\nu : (\Lambda_0 + x) \cap \Lambda \neq \emptyset} (\Lambda_0 + x).$$

Now for any finite Λ and $A \in \mathcal{D}_\Lambda$ the formal commutator $\delta(A) \equiv [H, A]$ is defined rigorously by

$$\delta(A) := [H_{\overline{\Lambda}}, A] = [H_{\Lambda,0}, A] + [\Phi_{\overline{\Lambda}}, A] \in \mathcal{B}(\mathcal{H}_{\overline{\Lambda}}).$$

This consistently defines $\delta(A) \in \mathcal{A}_\infty$ for all $A \in \mathcal{D}$. We adopt now the following

Definition 1. We say that a locally normal (i.e., given by a density matrix) state ω on $\mathcal{A}_\infty$ is *an infinite volume ground state of the Hamiltonian* H, if

$$\omega(A^* \delta(A)) \geqslant 0$$

for all $A \in \mathcal{D}$.

It will be convenient to consider also ground states in finite volumes; see ref. 4 for a discussion of possible definitions of equilibrium states in finite quantum systems with general boundary conditions.

For any Λ, let Λ^o be the part of Λ not interacting with the exterior

$$\Lambda^o := \Lambda \setminus \bigcup_{x \in \mathbb{Z}^\nu : (\Lambda_0 + x) \cap (\mathbb{Z}^\nu \setminus \Lambda) \neq \emptyset} (\Lambda_0 + x),$$

so that $\overline{\Lambda^o} \subset \Lambda$ and hence $\delta(A) \in \mathcal{B}(\mathcal{H}_\Lambda)$ if $A \in \mathcal{D}_{\Lambda^o}$.

Definition 2. For any finite Λ, we say that a normal state ω_Λ on $\mathcal{B}(\mathcal{H}_\Lambda)$ is *a finite volume ground state of the Hamiltonian* H, if

$$\omega(A^* \delta(A)) \geqslant 0 \tag{4}$$

for all $A \in \mathcal{D}_{\Lambda^o}$.

This definition plays a central role in the paper. It is possible to describe boundary conditions explicitly: (see Appendix A)

Clearly, a restriction of an infinite volume ground state to a finite volume is a finite volume ground state in the above sense; on the other hand, a weak*-limit of finite volume ground states is an infinite volume ground state. The state $\omega_{gs,\Lambda}$ of Theorem 1 satisfies Definition 2, because $\delta(A)=[H_\Lambda, A]$ for $A\in\mathcal{D}_{\Lambda^o}$. Therefore the state $\omega_{gs,\infty}$ of Theorem 1 is an infinite volume ground state in the sense of Definition 1.

Now we state the main result of the present paper as the following theorem.

Theorem 2. There exist positive constants c, c_1, c_2, depending only on the range Λ_0 and with $c_2<1$, such that if $\sup_x \|\phi_x\|<c$ then for any finite volume Λ, any two finite-volume ground states $\omega'_\Lambda, \omega''_\Lambda$ of the Hamiltonian H in Λ in the sense of Definition 2, and any $I\subset\Lambda$ one has

$$|\omega'_\Lambda(A)-\omega''_\Lambda(A)|\leqslant c_1^{|I|}c_2^{\operatorname{dist}(I,\mathbb{Z}^\nu\setminus\Lambda^o)}\|A\| \quad \text{if } A\in\mathcal{B}(\mathcal{H}_I). \tag{5}$$

Taking the limit $\Lambda\nearrow\mathbb{Z}^\nu$, we immediately obtain

Corollary. If $\sup_x\|\phi_x\|<c$ then the state $\omega_{gs,\infty}$ of Theorem 1 is a unique infinite-volume ground state of the Hamiltonian H.

Our proof uses some form of cluster expansions and is rather technical. Therefore, we first fix ideas in Section 2 by considering the simpler special case of perturbations preserving the ground state. The general case is treated in Section 3.

2. PERTURBATIONS PRESERVING THE GROUND STATE

In order to fix ideas, in this section, we prove Theorem 2 for the special case of perturbations not destroying the ground state. For any $\Lambda'\subset\mathbb{Z}^\nu$ we denote

$$\Omega_{\Lambda',0}:=\otimes_{x\in\Lambda'}\Omega_x.$$

We assume in this section that

$$\phi_x\Omega_{\Lambda_0+x,0}=0, \quad \forall x.$$

It follows then that for any Λ the vector $\Omega_{\Lambda,0}$ is an eigenvector of the operator H_Λ

$$H_\Lambda\Omega_{\Lambda,0}=0.$$

If $\sup_x \|\phi_x\|$ is small enough, namely $\sup_x \|\phi_x\| < 1/|\Lambda_0|$, then by the spectral gap condition $|\Lambda_0|^{-1} \sum_{y \in \Lambda_0 + x} h_y + \phi_x \geqslant 0$ and hence $H_\Lambda \geqslant 0$; so $\Omega_{\Lambda,0}$ is a ground state of H_Λ (i.e., $\Omega_\Lambda = \Omega_{\Lambda,0}$ in this case). We will see that the difference between this ground state and any other ground state ω in Λ in the sense of Definition 2 can be estimated as in (5). To this end it will suffice to use the local stability condition (4) with a family of single-site operators A. Let us denote

$$\mathcal{H}'_x = \mathcal{H}_x \ominus \Omega_x.$$

For any $u_x \in \mathcal{H}'_x$ we introduce the one-dimensional ("creation" or "spin raising") operator $\widehat{u}_x$ on $\mathcal{H}_x$ by

$$\widehat{u}_x v = (v, \Omega_x) u_x, \quad v \in \mathcal{H}_x. \tag{6}$$

(As usual, we can also consider $\widehat{u}_x$ as acting on $\mathcal{H}_{\Lambda'}$ for any $\Lambda' \ni x$.) The adjoint operator $\widehat{u}_x^*$ is then given by

$$\widehat{u}_x^* v = (v, u_x) \Omega_x, \quad v \in \mathcal{H}_x.$$

These adjoints are the operators which we will substitute into (4). This choice of trial operators is natural, since we expect that, when used on a state, they should generally lower its energy. In what follows, if S is a vector or a subspace in the Hilbert space $\mathcal{H}_{\Lambda'}$ for some Λ', then by P_S we shall denote the corresponding projector onto S in $\mathcal{H}_{\Lambda'}$; Λ' is not indicated in this notation, because which Λ' is meant in a particular situation will be clear from the form of the set S.

We claim now that using the local stability condition (4) with operators $\widehat{u}_x^*$ for some $x \in \Lambda^o$ yields the inequality

$$\omega(P_{\mathcal{H}'_x}) \leqslant \frac{1}{2|\Gamma_x|} \sum_{y \in \Gamma_x} \omega(P_{\mathcal{H}'_y}), \tag{7}$$

where Γ_x is a neighborhood of the site x

$$\Gamma_x = \bigcup_{y : \Lambda_0 + y \ni x} (\Lambda_0 + y).$$

We postpone the proof of this claim, and show now how it implies the desired estimate (5). Note first that by considering these inequalities for all $x \in \Lambda^o$ we can conclude that

$$\omega(P_{\mathcal{H}'_x}) \leqslant \left(\frac{1}{2}\right)^{\operatorname{dist}(x, \mathbb{Z}^\nu \setminus \Lambda^o)/\operatorname{diam}(\Lambda_0)}. \tag{8}$$

Indeed, for any $x \in \Lambda$ we have $\omega(P_{\mathcal{H}'_x}) \leqslant 1$. For any $x \in \Lambda^o$ using (7) we have $\omega(P_{\mathcal{H}'_x}) \leqslant 1/2$. For any x such that $\Gamma_x \subset \Lambda^o$ using (7) we have $\omega(P_{\mathcal{H}'_x}) \leqslant 1/4$, etc.: (8) is proved by iterations.

The exponential bound (8) for expectations of single-site projectors easily implies the estimate (5) for general local operators. Indeed, suppose that $A \in \mathcal{B}(\mathcal{H}_I)$ for some $I \subset \Lambda$. In order to estimate $\omega(A) - (A\Omega_{\Lambda,0}, \Omega_{\Lambda,0})$ we represent it as

$$\begin{aligned}\omega(A) - (A\Omega_{\Lambda,0}, \Omega_{\Lambda,0}) = {} & \omega(P_{\Omega_{I,0}} A P_{\mathcal{H}_I \ominus \Omega_{I,0}}) + \omega(P_{\mathcal{H}_I \ominus \Omega_{I,0}} A P_{\Omega_{I,0}}) \\ & + \omega(P_{\mathcal{H}_I \ominus \Omega_{I,0}} A P_{\mathcal{H}_I \ominus \Omega_{I,0}}) + (\omega(P_{\Omega_{I,0}} A P_{\Omega_{I,0}}) \\ & - (A\Omega_{\Lambda,0}, \Omega_{\Lambda,0})).\end{aligned} \tag{9}$$

Note that

$$P_{\mathcal{H}_{\Gamma_I} \ominus \Omega_{\Gamma_I,0}} \leqslant \sum_{y \in \Gamma_I} P_{\mathcal{H}'_y}. \tag{10}$$

Using Cauchy inequality, we estimate now the first term on the r.h.s. of (9)

$$\begin{aligned}|\omega(P_{\Omega_{I,0}} A P_{\mathcal{H}_I \ominus \Omega_{I,0}})| & \leqslant (\omega(P_{\mathcal{H}_I \ominus \Omega_{I,0}}))^{1/2} (\omega(P_{\Omega_{I,0}} A A^* P_{\Omega_{I,0}}))^{1/2} \\ & \leqslant (\omega(P_{\mathcal{H}_I \ominus \Omega_{I,0}}))^{1/2} \|A\| \\ & \leqslant |I|^{1/2} \left(\frac{1}{2}\right)^{\operatorname{dist}(I, \mathbb{Z}^\nu \setminus \Lambda^o)/2 \operatorname{diam}(\Lambda_0)} \|A\|,\end{aligned}$$

where in the last inequality we used estimates (10) and (8). The second and third terms in (9) are estimated in the same manner. For the final fourth term we have

$$\omega(P_{\Omega_{I,0}} A P_{\Omega_{I,0}}) - (A\Omega_{\Lambda,0}, \Omega_{\Lambda,0}) = \omega(B),$$

where $B = P_{\Omega_{I,0}} A P_{\Omega_{I,0}} - (A\Omega_{\Lambda,0}, \Omega_{\Lambda,0})\mathbf{1}$. Since $A \in \mathcal{B}(\mathcal{H}_I) \subset \mathcal{B}(\mathcal{H}_\Lambda)$, we have $P_{\Omega_{I,0}} A P_{\Omega_{I,0}} = (A\Omega_{\Lambda,0}, \Omega_{\Lambda,0}) P_{\Omega_{I,0}}$ and hence

$$B = -(A\Omega_{\Lambda,0}, \Omega_{\Lambda,0}) P_{\mathcal{H}_I \ominus \Omega_{I,0}}.$$

Now $\omega(B)$ can be estimated using inequalities (10) and (8) analogously to the other three terms in (9). That proves Theorem 2 in our present special case.

In the remaining part of the section we show how the inequality (7) follows from the local stability condition.

If $u_x \in \mathrm{Dom}\,(h_x) \cap \mathcal{H}'_x$, then

$$[h_x, \widehat{u}_x^*] = -\widehat{h_x u_x}^*.$$

It follows that:

$$\delta(\widehat{u}_x^*) = [h_x, \widehat{u}_x^*] + [\Phi_x, \widehat{u}_x^*] = -\widehat{h_x u_x}^* + [\Phi_x, \widehat{u}_x^*],$$

where

$$\Phi_x = \sum_{y:\Lambda_0 + y \ni x} \phi_y. \tag{11}$$

Condition (4) implies then that for $x \in \Lambda^o$

$$\omega(\widehat{u}_x \widehat{h_x u_x}^*) - \omega(\widehat{u}_x \Phi_x \widehat{u}_x^*) + \omega(\widehat{u}_x \widehat{u}_x^* \Phi_x) \leqslant 0. \tag{12}$$

We argue now that (12) implies

$$\omega(P_{\mathcal{H}'_x}) \leqslant \|\Phi_x\| \omega(P_{\mathcal{H}'_x}) - \omega(P_{\mathcal{H}'_x} \Phi_x). \tag{13}$$

Indeed, let us first consider the case $\dim \mathcal{H}_x < \infty$. For any unit vector $u_x \in \mathcal{H}'_x$ we have

$$\widehat{u}_x \widehat{u}_x^* = P_{u_x}$$

and

$$\widehat{u}_x \Phi_x \widehat{u}_x^* \leqslant \|\Phi_x\| \widehat{u}_x \widehat{u}_x^* = \|\Phi_x\| P_{u_x}.$$

Now if u_x is an eigenvector of h_x with an eigenvalue λ, then (12) implies

$$\lambda \omega(P_{u_x}) \leqslant \|\Phi_x\| \omega(P_{u_x}) - \omega(P_{u_x} \Phi_x). \tag{14}$$

Let $u_{x,1},\dots,u_{x,n}$ be an orthonormal eigenbasis for h_x in the subspace $\mathcal{H}'_x$. By the spectral gap condition, the corresponding eigenvalues are all not less than 1. Summing now the inequalities (14) for these eigenvectors and using the identity

$$\sum_k P_{u_{x,k}} = P_{\mathcal{H}'_x}, \tag{15}$$

we obtain (13).

In the case of infinite-dimensional $\mathcal{H}_x$ one can first approximate h_x by an operator with pure point spectrum. For a small positive ϵ consider the function $\kappa_\epsilon : a \mapsto \epsilon[a/\epsilon]$, where $[\cdot]$ is the integer part. Then $h_{x,\epsilon} := \kappa_\epsilon(h_x)$ has pure point spectrum, while the norm of the difference $g_{x,\epsilon} := h_{x,\epsilon} - h_x$ does not exceed ϵ. Let $u_{x,1}, u_{x,2},\dots$ be the eigenbasis for $h_{x,\epsilon}$; the corresponding eigenvalues then are all not less than $1-\epsilon$. For any $u_x \in \mathcal{H}'_x \cap \mathrm{Dom}\,(h_x)$,

$$\widehat{u}_x \widehat{h_x u_x}^* = \widehat{u}_x \widehat{h_{x,\epsilon} u_x}^* + P_{u_x} g_{x,\epsilon}.$$

Now if u_x is an eigenvector of $h_{x,\epsilon}$, then we have the inequality

$$(1-\epsilon)\omega(P_{u_x}) + \omega(P_{u_x} g_{x,\epsilon}) \leqslant \|\Phi_x\|\omega(P_{u_x}) - \omega(P_{u_x}\Phi_x), \tag{16}$$

analogous to the inequality (14). Again we have the identity (15), now with a strongly convergent series on the l.h.s. Summing the inequalities (16) for all eigenvectors and using at this point the normality of the state to take the limit, we obtain

$$(1-\epsilon)\omega(P_{\mathcal{H}'_x}) + \omega(P_{\mathcal{H}'_x} g_{x,\epsilon}) \leqslant \|\Phi_x\|\omega(P_{\mathcal{H}'_x}) - \omega(P_{\mathcal{H}'_x}\Phi_x).$$

Since $|\omega(P_{\mathcal{H}'_x} g_{x,\epsilon})| \leqslant \epsilon$, letting $\epsilon \to 0$ proves (13).

By definition (11) of Φ_x its norm does not exceed $|\Lambda_0| \sup_x \|\phi_x\|$. Hence it follows from (13) that:

$$(1 - |\Lambda_0| \sup_x \|\phi_x\|)\omega(P_{\mathcal{H}'_x}) \leqslant |\omega(P_{\mathcal{H}'_x}\Phi_x)| \leqslant (\omega(P_{\mathcal{H}'_x}))^{1/2}(\omega(\Phi_x^2))^{1/2}$$

by the Cauchy inequality. Therefore,

$$\omega(P_{\mathcal{H}'_x}) \leqslant (1 - |\Lambda_0| \sup_x \|\phi_x\|)^{-2}\omega(\Phi_x^2). \tag{17}$$

Recall that by our assumption in this section $\Phi_x \Omega_{\Gamma_x,0} = 0$ and hence

$$\Phi_x^2 = P_{\mathcal{H}_{\Gamma_x} \ominus \Omega_{\Gamma_x,0}} \Phi_x^2 P_{\mathcal{H}_{\Gamma_x} \ominus \Omega_{\Gamma_x,0}} \leqslant \|\Phi_x^2\| P_{\mathcal{H}_{\Gamma_x} \ominus \Omega_{\Gamma_x,0}}.$$

It follows then from (17) and (10) that

$$\omega(P_{\mathcal{H}'_x}) \leqslant \left(\frac{|\Lambda_0| \sup_x \|\phi_x\|}{1 - |\Lambda_0| \sup_x \|\phi_x\|} \right)^2 \sum_{y \in \Gamma_x} \omega(P_{\mathcal{H}'_y}),$$

which implies the desired inequality (7), if $\sup_x \|\phi_x\|$ is sufficiently small.

3. PROOF OF THEOREM 2 IN THE GENERAL CASE

We will prove Theorem 2 in its general form by a suitable generalization of the argument used in Section 2. As before, we can obtain one ground state in Λ simply by considering the Hamiltonian H_Λ with empty boundary conditions. As stated in Theorem 1, this Hamiltonian has a non-degenerate ground state, corresponding to the vector Ω_Λ, which can in principle be used for comparison with other, general ground states. However, the new element now is that the state $(\cdot \Omega_\Lambda, \Omega_\Lambda)$ is no longer a product of single site vector states, and hence the deviation of a general ground-state from this special state cannot be measured simply by expectations of single-site projectors. This difficulty can be handled by appropriate transformations in the local algebra. The ground-state vector Ω_Λ and the product vector $\Omega_{\Lambda,0}$ can be obtained from one another using a suitable "dressing transformation". By using related automorphisms of the algebra one can in a sense reduce the problem to the "non-entangled" case of Section 2. The deviation from $(\cdot \Omega_\Lambda, \Omega_\Lambda)$ will now be measured by some quasi-local positive perturbations Q_x of the projectors $P_{\mathcal{H}'_x}$ (see Lemma 2). The statement of Theorem 2 will then follow from the smallness of those deviations by some sort of cluster expansion.

In this section, we adopt for brevity the following convention. We will denote by c and ϵ various (generally different in different formulas) positive constants, which do not depend on the volume Λ, though may depend on the interaction range Λ_0. We write ϵ if this constant can be chosen arbitrarily small by choosing $\sup_x \|\phi_x\|$ small enough; on the other hand the constant c is typically greater than 1 and does not depend on $\sup_x \|\phi_x\|$. We will omit some standard cumbersome calculations typical of cluster expansions.

We begin with some preliminaries concerning the structure of the ground state vector Ω_Λ and related "dressing transformations". An important role in our proof is played by the following ansatz for this vector (ref. 10, also see refs. 1 and 7).

For any $I \subset \Lambda$ set

$$\mathcal{H}'_I = \otimes_{x\in I}\mathcal{H}'_x,$$

where $\mathcal{H}'_x = \mathcal{H}_x \ominus \Omega_x$ (with $\mathcal{H}'_\emptyset \equiv \mathbb{C}$). It follows that:

$$\mathcal{H}_\Lambda \ominus \Omega_{\Lambda,0} = \bigoplus_{\emptyset\neq I\subset\Lambda} \mathcal{H}'_I \otimes \Omega_{\Lambda\setminus I,0}. \tag{18}$$

Vectors from $\mathcal{H}'_I$ will be denoted by u_I, v_I, etc. For each $u_I \in \mathcal{H}'_I$ we introduce a "creation" operator $\widehat{u}_I$ in $\mathcal{H}_I$ by

$$\widehat{u}_I v = (v, \Omega_{I,0})u_I, \quad v \in \mathcal{H}_I,$$

like in (6) (with $\widehat{u}_\emptyset$ a scalar operator). A useful property of these operators is that for any I, J and u_I, v_J

$$\widehat{u}_I\widehat{v}_J = \begin{cases} 0 & \text{if } I\cap J \neq \emptyset, \\ \widehat{u_I\otimes v_J} & \text{if } I\cap J = \emptyset. \end{cases}$$

In particular, they commute. For any $v \in \mathcal{H}_\Lambda$ such that $(v, \Omega_{\Lambda,0}) = 1$ there exists a unique collection $\{v_I \in \mathcal{H}'_I\}_{\emptyset\neq I\subset\Lambda}$ such that $\exp\big(\sum_{\emptyset\neq I\subset\Lambda}\widehat{v}_I\big)\Omega_{\Lambda,0}=v$ (these v_I can be obtained by truncation from components of v appearing in the decomposition (18)). Let $\widetilde{\Omega}_\Lambda$ be the ground state vector of H_Λ normalized so that $(\widetilde{\Omega}_\Lambda, \Omega_{\Lambda,0}) = 1$ (i.e., $\widetilde{\Omega}_\Lambda := \Omega_\Lambda/(\Omega_\Lambda, \Omega_{\Lambda,0})$). Initially the non-degeneracy of the ground state and its non-orthogonality to $\Omega_{\Lambda,0}$ is clear from the usual finite-volume perturbation theory for sufficiently weak perturbations in each particular volume, and it can be shown that the estimate for the perturbation, which ensures this property, can actually be chosen uniform in the volume. Let $\{v_I^{(gs)} \in \mathcal{H}'_I\}_{\emptyset\neq I\subset\Lambda}$ be the corresponding collection such that

$$\widetilde{\Omega}_\Lambda = \exp\Big(\sum_{\emptyset\neq I\subset\Lambda}\widehat{v}_I^{(gs)}\Big)\Omega_{\Lambda,0}.$$

Lemma 1. (ref. 10) For any $\epsilon > 0$, if $\sup_x \|\phi_x\|$ is sufficiently small then

$$\max_{x\in\Lambda} \sum_{I\subset\Lambda : x\in I} \|v_I^{(gs)}\| \epsilon^{-(d_I+1)} \leqslant 1, \tag{19}$$

where d_I is the minimal length of a connected graph containing I.

In what follows we will use the (non-involutive and non-norm-preserving) automorphism α on $\mathcal{B}(\mathcal{H}_\Lambda)$ given by

$$\alpha(A) = \exp\big(\sum_{\emptyset\neq I\subset\Lambda} \widehat{v}_I^{(gs)}\big) A \exp\big(-\sum_{\emptyset\neq I\subset\Lambda} \widehat{v}_I^{(gs)}\big). \tag{20}$$

We denote the inverse automorphism by α_- and also denote $\alpha^*(\cdot) \equiv (\alpha(\cdot))^*$. A useful property of α (and similarly α_-, α^*) is that it can be expanded into a commutator series

$$\alpha(A) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \sum_{\emptyset\neq I_1,\ldots,I_s\subset\Lambda} [\ldots[A, \widehat{v}_{I_1}^{(gs)}], \ldots, \widehat{v}_{I_s}^{(gs)}]. \tag{21}$$

If $A \in \mathcal{B}(\mathcal{H}_I)$ for some $I \subset \Lambda$, then the summation in the above series can be restricted to those terms, where $I_k \cap I \neq \emptyset$ for all k, because the remaining terms vanish due to the commutativity of the creation operators. In particular, using this expansion and Lemma 1, it is easy to obtain an estimate of the form

$$\|\alpha(A)\| \leqslant c^{|I|} \|A\|, \quad A \in \mathcal{B}(\mathcal{H}_I) \tag{22}$$

with some $c > 1$.

Commutativity of creation operators implies for any u_I

$$\alpha(\widehat{u}_I) = \alpha_-(\widehat{u}_I) = \widehat{u}_I. \tag{23}$$

Let $\widetilde{H}_\Lambda = H_\Lambda - E_\Lambda \mathbf{1}$, where E_Λ is the ground state energy of H_Λ, so that

$$\widetilde{H}_\Lambda \Omega_\Lambda = 0. \tag{24}$$

There is a convenient commutator expansion for $\alpha_-(\widetilde{H}_\Lambda) - H_{\Lambda,0}$. Note first that for any $u_I \in \mathcal{H}'_I \cap \mathrm{Dom}\,(H_{I,0}), I \subset \Lambda$,

$$[H_{\Lambda,0}, \widehat{u}_I] = \widehat{H_{I,0}u_I}. \tag{25}$$

Using (23), (24), it follows that:

$$\begin{aligned}
\alpha_-(\widetilde{H}_\Lambda)\widehat{u}_I\Omega_{\Lambda,0} &= \alpha_-(\widetilde{H}_\Lambda\widehat{u}_I)\Omega_{\Lambda,0}\\
&= \alpha_-([\widetilde{H}_\Lambda, \widehat{u}_I])\Omega_{\Lambda,0}\\
&= \alpha_-([H_\Lambda, \widehat{u}_I])\Omega_{\Lambda,0}\\
&= \alpha_-([H_{\Lambda,0}, \widehat{u}_I])\Omega_{\Lambda,0} + \alpha_-([\Phi_\Lambda, \widehat{u}_I])\Omega_{\Lambda,0}\\
&= \widehat{H_{I,0}u_I}\Omega_{\Lambda,0} + \alpha_-([\Phi_\Lambda, \widehat{u}_I])\Omega_{\Lambda,0}.
\end{aligned}$$

Hence $\alpha_-(\widetilde{H}_\Lambda) - H_{\Lambda,0}$ is a bounded operator such that

$$(\alpha_-(\widetilde{H}_\Lambda) - H_{\Lambda,0})\widehat{u}_I\Omega_{\Lambda,0} = \alpha_-([\Phi_\Lambda, \widehat{u}_I])\Omega_{\Lambda,0}$$

$$= \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{\substack{x:\Lambda_0+x\subset\Lambda\\ (\Lambda_0+x)\cap I\neq\emptyset}}\ \sum_{\substack{\emptyset\neq I_1,\dots,I_s\subset\Lambda\\ I_k\cap(\Lambda_0+x)\neq\emptyset\,\forall k}} [\dots[[\phi_x, \widehat{u}_I], \widehat{v}^{(gs)}_{I_1}], \dots, \widehat{v}^{(gs)}_{I_s}]\Omega_{\Lambda,0}. \tag{26}$$

After these preliminaries we begin the proof of Theorem 2. First we establish an analog of the estimate (8). Let us fix some finite volume Λ with the interior Λ^o and consider the automorphism $\alpha : \mathcal{B}(\mathcal{H}_{\Lambda^o}) \to \mathcal{B}(\mathcal{H}_{\Lambda^o})$ defined as in (20), but with respect to the set Λ^o (and w.r.t. the corresponding Hamiltonian H_{Λ^o} and its ground state Ω_{Λ^o}.) Let $u_{x,1}, u_{x,2}, \dots$ be an orthonormal basis in $\mathcal{H}'_x$ for some $x \in \Lambda^o$ and set

$$Q_x = \sum_k \alpha^*(\widehat{u}^*_{x,k})\alpha(\widehat{u}^*_{x,k}).$$

It is easy to see that Q_x does not depend on the choice of the basis. If $\dim \mathcal{H}'_x = \infty$, then the r.h.s. is an infinite sum of non-negative operators; it is strongly convergent because

$$\sum_k A^*\widehat{u}_{x,k}B^*B\widehat{u}^*_{x,k}A \leqslant \|B^*B\| \sum_k A^*\widehat{u}_{x,k}\widehat{u}^*_{x,k}A = \|B^*B\|A^*P_{\mathcal{H}'_x}A.$$

We will prove now

Lemma 2. If ω is a ground state in Λ then

$$\omega(Q_x) \leqslant \epsilon^{\operatorname{dist}(x, \mathbb{Z}^\nu \setminus \Lambda^o)}. \tag{27}$$

Proof. We follow the argument in Section 2, but now use the local stability condition (4) with operators of the form $\alpha(\widehat{u}_x^*)$, $x \in \Lambda^o$, instead of $\widehat{u}_x^*$

$$\begin{aligned}&\omega(\alpha^*(\widehat{u}_x^*)[H_\Lambda, \alpha(\widehat{u}_x^*)])\\ &\quad = \omega(\alpha^*(\widehat{u}_x^*)[H_{\Lambda^o}, \alpha(\widehat{u}_x^*)]) + \omega\Big(\alpha^*(\widehat{u}_x^*)\Big[\sum_{\substack{y:\Lambda_0+y\subset\Lambda\\ \Lambda_0+y\notin\Lambda^o}} \phi_y, \alpha(\widehat{u}_x^*)\Big]\Big) \geqslant 0,\end{aligned} \tag{28}$$

where we have isolated the boundary interaction. Next we write

$$\begin{aligned}[H_{\Lambda^o}, \alpha(\widehat{u}_x^*)] &= [\widetilde{H}_{\Lambda^o}, \alpha(\widehat{u}_x^*)]\\ &= \alpha([\alpha_-(\widetilde{H}_{\Lambda^o}), \widehat{u}_x^*])\\ &= \alpha([H_{\Lambda^o,0}, \widehat{u}_x^*]) + \alpha([\alpha_-(\widetilde{H}_{\Lambda^o}) - H_{\Lambda^o,0}, \widehat{u}_x^*])\\ &= -\alpha\Big(\widehat{h_x u_x}^*\Big) + \alpha([\alpha_-(\widetilde{H}_{\Lambda^o}) - H_{\Lambda^o,0}, \widehat{u}_x^*]).\end{aligned}$$

Let

$$\alpha_-(\widetilde{H}_{\Lambda^o}) - H_{\Lambda^o,0} = \sum_{J\subset\Lambda^o} \Psi_J,$$

where the operator Ψ_J is defined by the sum of those terms in the expansion (26) for which $(\Lambda_0 + x) \cup (\cup_k I_k) = J$. Note that $[\Psi_J, \widehat{u}_x^*] = 0$ if $x \notin J$. Let

$$\Psi_x = \sum_{J\ni x} \Psi_J$$

it follows then that:

$$[H_{\Lambda^o}, \alpha(\widehat{u}_x^*)] = -\alpha\Big(\widehat{h_x u_x}^*\Big) + [\alpha(\Psi_x), \alpha(\widehat{u}_x^*)].$$

Lemma 1 implies

$$\|\Psi_J\| \leqslant \epsilon^{d_J} \tag{29}$$

and this in turn implies $\|\alpha(\Psi_x)\| \leqslant \epsilon$ by (22). Now one can proceed in the same manner as in Section 2 by choosing orthonormal eigenvectors of h_x and adding corresponding inequalities (28). This yields

$$\omega(Q_x) \leqslant 2(\omega(Q_x))^{1/2}(\omega(\alpha^*(\Psi_x)\alpha(\Psi_x)))^{1/2} + \epsilon^{\operatorname{dist}(x,\mathbb{Z}^\nu\setminus\Lambda^o)}, \tag{30}$$

because

$$\begin{aligned}
\Big|\sum_k \omega(\alpha^*(\widehat{u}^*_{x,k})\alpha(\widehat{u}^*_{x,k})\alpha(\Psi_x))\Big| &= |\omega(Q_x\alpha(\Psi_x))| \\
&\leqslant (\omega(Q_x))^{1/2}\Big(\omega(\alpha^*(\Psi_x)\alpha(\Psi_x))\Big)^{1/2}, \\
\Big|\sum_k \omega(\alpha^*(\widehat{u}^*_{x,k})\alpha(\Psi_x)\alpha(\widehat{u}^*_{x,k}))\Big| &\leqslant \sum_k \|\alpha(\Psi_x)\|\omega(\alpha^*(\widehat{u}^*_{x,k})\alpha(\widehat{u}^*_{x,k})) \\
&\leqslant \epsilon\omega(Q_x)
\end{aligned}$$

and because by similar estimates and Lemma 1

$$\Big|\sum_k \omega\Big(\alpha^*(\widehat{u}^*_{x,k})\Big[\sum_{\substack{y:\Lambda_0+y\subset\Lambda \\ \Lambda_0+y\notin\Lambda^o}} \phi_y, \alpha(\widehat{u}^*_{x,k})\Big]\Big)\Big| \leqslant \epsilon^{\operatorname{dist}(x,\mathbb{Z}^\nu\setminus\Lambda^o)}.$$

Inequality (30) implies

$$\omega(Q_x) \leqslant 4\Big(\omega(\alpha^*(\Psi_x)\alpha(\Psi_x)) + \epsilon^{\operatorname{dist}(x,\mathbb{Z}^\nu\setminus\Lambda^o)}\Big). \tag{31}$$

By Cauchy inequality

$$\begin{aligned}
\omega(\alpha^*(\Psi_x)\alpha(\Psi_x)) &\leqslant \sum_{I\ni x, J\ni x} |\omega(\alpha^*(\Psi_I)\alpha(\Psi_J))| \\
&\leqslant \sum_{I\ni x, J\ni x} \Big(\omega(\alpha^*(\Psi_I)\alpha(\Psi_I))\Big)^{1/2}\Big(\omega(\alpha^*(\Psi_J)\alpha(\Psi_J))\Big)^{1/2} \\
&= \Big(\sum_{I\ni x}\Big(\omega(\alpha^*(\Psi_I)\alpha(\Psi_I))\Big)^{1/2}\Big)^2.
\end{aligned} \tag{32}$$

We will now estimate this expression using the following Lemma.

Lemma 3. There exists a constant c such that if $A \in \mathcal{B}(\mathcal{H}_I)$ and $A\Omega_{I,0} = 0$ for some non-empty $I \subset \Lambda^o$ then

$$\alpha^*(A)\alpha(A) \leqslant \|A\|^2 c^{|I|} \sum_{x \in I} Q_x.$$

Proof. Without loss of generality assume that $\|A\| \leqslant 1$. We have to show that $B \geqslant 0$, where

$$B = \sum_{x \in I} Q_x - c^{-|I|}\alpha^*(A)\alpha(A).$$

This is equivalent to showing that $\alpha_-(B) - \lambda$ is invertible for any negative λ, because $\alpha_-(B)$ is similar to B. To this end using (21) we expand

$$\begin{aligned}\alpha_-(Q_x) &= \sum_k \alpha_-(\alpha^*(\widehat{u}^*_{x,k}))\widehat{u}^*_{x,k} \\ &= \sum_k \sum_{p,q=0}^{\infty} \frac{1}{p!q!} \sum_{\substack{I_1,\dots,I_p: \\ I_s \ni x\, \forall s}} \sum_{\substack{J_1,\dots,J_q: \\ \forall t\, \exists s:\, J_t \cap I_s \neq \emptyset}} \\ &\quad [\dots[[\dots[\widehat{u}_{x,k}, \widehat{v}^{(gs)*}_{I_1}], \dots, \widehat{v}^{(gs)*}_{I_p}], \widehat{v}^{(gs)}_{J_1}], \dots, \widehat{v}^{(gs)}_{J_q}]\widehat{u}^*_{x,k}.\end{aligned}$$

It is easy to see from this expression and Lemma 1 that

$$\alpha_-(Q_x) = \Big(\mathbf{1} + \sum_{J \ni x} R_{x,J}\Big) P_{\mathcal{H}'_x},$$

where $R_{x,J} \in \mathcal{B}(\mathcal{H}_J)$ and $\|R_{x,J}\| \leqslant \epsilon^{d_J+1}$. Let

$$\alpha_-(B) = L + V,$$

where

$$\begin{aligned} L &= \sum_{x \in I} P_{\mathcal{H}'_x}, \quad V = V_1 + V_2, \\ V_1 &= \sum_{x \in I} \sum_{J \ni x} R_{x,J} P_{\mathcal{H}'_x}, \quad V_2 = -c^{-|I|}\alpha_-(\alpha^*(A))A. \end{aligned}$$

We will prove that $\alpha_-(B)-\lambda$ is invertible using the resolvent identity

$$(\alpha_-(B)-\lambda)^{-1}=\sum_{k=0}^{\infty}(-1)^k(L-\lambda)^{-1}(V(L-\lambda)^{-1})^k.$$

We will introduce a new norm $\|\cdot\|_I$ in $\mathcal{H}_{\Lambda^o}$, which is equivalent to the usual norm $\|\cdot\|$ but such that $\|V(L-\lambda)^{-1}\|_I<1$, which ensures that the above series converges and the resolvent exists. To this end we decompose

$$\mathcal{H}_{\Lambda^o}=\oplus_{K\subset I}\mathcal{G}_K^{(I)},$$

where

$$\mathcal{G}_K^{(I)}=\mathcal{H}'_K\otimes\Omega_{I\setminus K,0}\otimes\mathcal{H}_{\Lambda^o\setminus I}$$

and set

$$\|v\|_I=\sum_{K\subset I}\|v_{\mathcal{G}_K^{(I)}}\|,$$

where $v_{\mathcal{G}_K^{(I)}}$ is the projection of v to $\mathcal{G}_K^{(I)}$. Note that $L|_{\mathcal{G}_K^{(I)}}=|K|\mathbf{1}$ and hence

$$\|V(L-\lambda)^{-1}\|_I=\sup_{K\subset I}\sup_{\substack{v\in\mathcal{G}_K^{(I)}\\ \|v\|=1}}\|V(L-\lambda)^{-1}v\|_I=\sup_{K\subset I}\sup_{\substack{v\in\mathcal{G}_K^{(I)}\\ \|v\|=1}}\frac{\|Vv\|_I}{|K|-\lambda}. \quad (33)$$

Note first that if $K=\emptyset$ then $V|_{\mathcal{G}_\emptyset^{(I)}}=0$ by assumption on A, hence $\sup_{K\subset I}$ in the above formula can be restricted to sup over non-empty K. Next, similarly to (22) one can prove

$$\|\alpha_-(\alpha^*(A))\|\leqslant c'^{|I|}\|A\|,\quad A\in\mathcal{B}(\mathcal{H}_I)$$

with some constant c' and hence, if $\|A\|\leqslant 1$,

$$\|V_2\|_I=\|c^{-|I|}\alpha_-(\alpha^*(A))A\|_I\leqslant 2^{|I|}\|c^{-|I|}\alpha_-(\alpha^*(A))A\|\leqslant(2c'/c)^{|I|}\leqslant 1/3$$

if c is chosen larger than $6c'$. Next, if $v\in\mathcal{G}_K^{(I)}$ then

$$V_1v=\sum_{x\in K}\sum_{J\ni x}R_{x,J}v.$$

It's easy to see that $\|R_{x,J}\|_I \leqslant 4^{|J|}\|R_{x,J}\|$ (regardless of I) because $R_{x,J} \in \mathcal{B}(\mathcal{H}_J)$. Therefore for $v \in \mathcal{G}_K^{(I)}$

$$\|V_1 v\|_I \leqslant |K| \sum_{J \ni 0} (4\epsilon)^{d_J+1} \|v\|,$$

which is not greater than $|K|\|v\|/3$ for ϵ small enough. So we see that for a negative λ the r.h.s. of (33) does not exceed

$$\sup_{k=1,2,\ldots} (k/3+1/3)/k = 2/3,$$

which completes the proof. ▮

Using inequalities (31) and (32), Lemma 3 and estimate (29), one obtains an estimate of the form

$$\omega(Q_x) \leqslant \sum_{y \in \Lambda^o} \epsilon^{|x-y|+1} \omega(Q_y) + \epsilon^{\operatorname{dist}(x, \mathbb{Z}^\nu \setminus \Lambda^o)}.$$

Our claim (27) then follows by iterations, as in Section 2. ▮

Now we show how Lemma 2 can be used to prove the desired estimate (5) of Theorem 2. First we claim that it suffices to establish it for creation operators $\widehat{u}_I$. To see this take any operator $A \in \mathcal{B}(\mathcal{H}_I)$ and represent it as $\alpha(\alpha_-(A))$. Using the commutator expansion and Lemma 1, it follows that:

$$\alpha_-(A) = \sum_{J \subset \Lambda^o} B_J, \quad B_J \in \mathcal{B}(\mathcal{H}_J),$$

where

$$\sum_J \|B_J\| \epsilon^{-d_{J,I}} \leqslant c^{|I|} \|A\|, \tag{34}$$

where $d_{J,I}$ is the minimal length of a (not generally connected) graph containing J and connecting any point of J with some point of I. Now, for any J there is a (unique) expansion

$$B_J = \sum_{K \subset J} \widehat{u}_K^{(B_J)} + \widetilde{B}_J, \text{ where } \quad \widetilde{B}_J \Omega_{J,0} = 0 \tag{35}$$

the vectors $u_K^{(B_J)} \in \mathcal{H}'_K$ are determined from the expansion

$$B_J \Omega_{J,0} = \sum_{K \subset J} u_K^{(B_J)} \otimes \Omega_{J \setminus K,0}.$$

Since $\|u_K^{(B_J)}\| \leqslant \|B_J\|$, we have

$$\|\widetilde{B}_J\| \leqslant (2^{|J|}+1)\|B_J\|. \tag{36}$$

By Lemmas 3 and 2

$$|\omega(\alpha(\widetilde{B}_J))| \leqslant (\omega(\alpha^*(\widetilde{B}_J)\alpha(\widetilde{B}_J)))^{1/2} \leqslant \|\widetilde{B}_J\| c^{|J|/2} |J|^{1/2} \epsilon^{\operatorname{dist}(\mathbb{Z}^\nu \setminus \Lambda^o, J)/2}.$$

Now using the estimates (34), (36) and the inequalities

$$\operatorname{dist}(\mathbb{Z}^\nu \setminus \Lambda^o, J) \geqslant \operatorname{dist}(\mathbb{Z}^\nu \setminus \Lambda^o, I) - d_{J;I}, \quad |J| \leqslant |I| + d_{J;I}, \tag{37}$$

one finds an estimate of the form

$$\Big|\omega\Big(\sum_{J \subset \Lambda^o} \alpha(\widetilde{B}_J)\Big)\Big| \leqslant c^{|I|} \epsilon^{\operatorname{dist}(\mathbb{Z}^\nu \setminus \Lambda^o, I)} \|A\|. \tag{38}$$

Now if ω', ω'' are two different ground states in Λ, then

$$\begin{aligned}
|\omega'(A) - \omega''(A)| &\leqslant \Big| \sum_{J \subset \Lambda^o} \sum_{K \subset J} \omega'\big(\alpha(\widehat{u}_K^{(B_J)})\big) - \omega''\big(\alpha(\widehat{u}_K^{(B_J)})\big)\Big| \\
&\quad + \Big|\omega'\Big(\sum_{J \subset \Lambda^o} \alpha(\widetilde{B}_J)\Big)\Big| + \Big|\omega''\Big(\sum_{J \subset \Lambda^o} \alpha(\widetilde{B}_J)\Big)\Big| \\
&\leqslant \sum_{J \subset \Lambda^o} \sum_{K \subset J} \big|\omega'(\widehat{u}_K^{(B_J)}) - \omega''(\widehat{u}_K^{(B_J)})\big| + 2c^{|I|} \epsilon^{\operatorname{dist}(\mathbb{Z}^\nu \setminus \Lambda^o, I)} \|A\|.
\end{aligned}$$

Using again (34) and (37), we see that if Theorem 2 holds for operators $\widehat{u}_I$, then it holds for all operators (perhaps with a larger constant c_1).

In order to show that Theorem 2 holds for these special operators we develop a system of linear equations for their averages. Let $\mathcal{K}$ be the linear space of operators of the form

$$\widehat{\mathbf{u}} = \sum_{\emptyset \neq I \subset \Lambda^o} \widehat{u}_I.$$

We will obtain below an equation of the form

$$\widehat{\mathbf{u}} = \mathcal{T}_1\widehat{\mathbf{u}} + \mathcal{T}_2\widehat{\mathbf{u}} + \mathcal{T}_3\widehat{\mathbf{u}} \tag{39}$$

with some linear operators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, where $\mathcal{T}_1 : \mathcal{K} \to \mathcal{K}$ is a contraction, $\mathcal{T}_2\widehat{\mathbf{u}}$ is a scalar operator, and $\mathcal{T}_3\widehat{\mathbf{u}}$ has small ω-average. To this end, take some nonempty $I \subset \Lambda^o$, fix some $x \in I$ and let $I' = I \setminus \{x\}$. Let $u_I \in \mathcal{H}'_I$. Choose an orthonormal basis $u_{x,1}, u_{x,2}, \ldots$ in $\mathcal{H}'_x$ and consider the expansion

$$u_I = \sum_k u_{x,k} \otimes u_{I',k},$$

where $u_{I',k} \in \mathcal{H}'_{I'}$ and $\sum_k \|u_{I',k}\|^2 = \|u_I\|^2$. Note that by Cauchy inequality

$$\begin{aligned}\Big|\sum_k \omega(\alpha^*(\widehat{u}^*_{x,k})\widehat{u}_{I',k})\Big| &\leqslant \Big(\sum_k \omega(\alpha^*(\widehat{u}^*_{x,k})\alpha(\widehat{u}^*_{x,k}))\Big)^{1/2}\Big(\sum_k \omega(\widehat{u}^*_{I',k}\widehat{u}_{I',k})\Big)^{1/2}\\ &= (\omega(Q_x))^{1/2}\|u_I\|(\omega(P_{\Omega_{I',0}}))^{1/2} \leqslant \|u_I\|\epsilon^{\,\mathrm{dist}\,(x,\mathbb{Z}^\nu\setminus\Lambda^o)}.\end{aligned} \tag{40}$$

On the other hand we write

$$\sum_k \alpha^*(\widehat{u}^*_{x,k})\widehat{u}_{I',k} = \alpha\Big(\sum_k \alpha_-(\alpha^*(\widehat{u}^*_{x,k}))\widehat{u}_{I',k}\Big) \tag{41}$$

and then expand $\alpha_-(\alpha^*(\widehat{u}^*_{x,k}))$ into a commutator series. As a result we obtain on the r.h.s. an expression of the form $\alpha(\ldots)$ with the leading term in brackets being $\sum_k \widehat{u}_{x,k}\widehat{u}_{I',k} = \widehat{u}_I$, other terms being of the form

$$\sum_k A\widehat{u}_{x,k}B\widehat{u}_{I',k} \tag{42}$$

with some operators A, B. It is easy to see that for any A, B the series (42) strongly converges to an operator C with

$$\|C\| \leqslant \|A\|\|B\|\|u_I\|. \tag{43}$$

If $A \in \mathcal{B}(\mathcal{H}_J), B \in \mathcal{B}(\mathcal{H}_K)$, then $C \in \mathcal{B}(\mathcal{H}_{I\cup J\cup K})$. We consider next for any such C the expansion

$$C = \sum_{L \subset I\cup J\cup K} \widehat{v}_L^{(C)} + \widetilde{C} \tag{44}$$

with $\widetilde{C}\Omega_{I\cup J\cup K,0}=0$. Note that here

$$v_L^{(C)}=0 \text{ if } I\setminus(J\cup K)\not\subset L. \tag{45}$$

After these expansions the r.h.s. of (41) takes on the form

$$\widehat{u}_I+\sum_{J\subset\Lambda^o}\widehat{v}_J^{(u_I)}+\alpha(\widetilde{C}(u_I)),$$

where the second term is obtained as the sum of all creation operators appearing in (44) for all terms in the commutator series (we use here the identity $\alpha(\widehat{v}_I)=\widehat{v}_I$), and $\widetilde{C}(u_I)$ is the contribution from the operators $\widetilde{C}$. We define now

$$\begin{aligned}\mathcal{T}_1\widehat{u}_I &= -\sum_{\emptyset\neq J\subset\Lambda^o}\widehat{v}_J^{(u_I)}, \quad \mathcal{T}_2\widehat{u}_I=-\widehat{v}_\emptyset^{(u_I)},\\ \mathcal{T}_3\widehat{u}_I &= \widehat{u}_I-\mathcal{T}_1\widehat{u}_I-\mathcal{T}_2\widehat{u}_I=\sum_k\alpha^*(\widehat{u}_{x,k}^*)\widehat{u}_{I',k}-\alpha(\widetilde{C}(u_I))\end{aligned}$$

and extend to $\mathcal{K}$ by linearity. Using the commutator expansion, Lemma 1 and (43), (45), we have

$$\|v_{(I\setminus J)\cup K}^{(u_I)}\|\leqslant\|u_I\|\epsilon^{d_{J\cup K\cup\{x\}}} \tag{46}$$

and also, like in (38),

$$\left|\omega\big(\alpha(\widetilde{C}(u_I))\big)\right|\leqslant c^{|I|}\epsilon^{\operatorname{dist}(\mathbb{Z}^\nu\setminus\Lambda^o,I)}\|u_I\|. \tag{47}$$

Suppose that $\mathcal{K}$ is equipped with the norm

$$\|\sum_{\emptyset\neq I\subset\Lambda^o}\widehat{u}_I\|_{\mathcal{K}}=\sum_{\emptyset\neq I\subset\Lambda^o}\|u_I\|c^{|I|}\epsilon^{\operatorname{dist}(\mathbb{Z}^\nu\setminus\Lambda^o,I)}.$$

Using the inequality (46) with ϵ small enough, one finds that $\|\mathcal{T}_1\|_{\mathcal{K}}<1/2$. Now for any ground-state ω we have

$$\omega(\widehat{\mathbf{u}})=\omega(\mathcal{T}_1\widehat{\mathbf{u}})+\omega(\mathcal{T}_2\widehat{\mathbf{u}})+\omega(\mathcal{T}_3\widehat{\mathbf{u}}).$$

Since $\mathcal{T}_2\widehat{\mathbf{u}}$ is a scalar operator, the second term in the r.h.s. is the same for all states. Suppose that we have two ground states ω', ω'', then

$$\omega'(\widehat{\mathbf{u}}) - \omega''(\widehat{\mathbf{u}}) = \omega'(\mathcal{T}_1\widehat{\mathbf{u}}) - \omega''(\mathcal{T}_1\widehat{\mathbf{u}}) + \omega'(\mathcal{T}_3\widehat{\mathbf{u}}) - \omega''(\mathcal{T}_3\widehat{\mathbf{u}}).$$

If $\mathbf{f} \in \mathcal{K}^*$ is a continuous linear functional on $\mathcal{K}$, then its norm is given by

$$\|\mathbf{f}\|_{\mathcal{K}^*} = \max_{\emptyset \neq I \subset \Lambda^o} \sup_{\|u_I\|=1} |\mathbf{f}(\widehat{u}_I)| c^{-|I|} \epsilon^{-\operatorname{dist}(\mathbb{Z}^\nu \setminus \Lambda^o, I)}.$$

Let $\mathbf{f}', \mathbf{f}''$ be the restrictions of the ground-states ω', ω'' to $\mathcal{K}$, and $\mathbf{g}', \mathbf{g}''$ be given by $\mathbf{g}'(\widehat{\mathbf{u}}) = \omega'(\mathcal{T}_3\widehat{\mathbf{u}}), \mathbf{g}''(\widehat{\mathbf{u}}) = \omega''(\mathcal{T}_3\widehat{\mathbf{u}})$. It follows that:

$$\mathbf{f}' - \mathbf{f}'' = \mathcal{T}_1^*(\mathbf{f}' - \mathbf{f}'') + \mathbf{g}' - \mathbf{g}'',$$

where $\mathcal{T}_1^*$ is the adjoint of $\mathcal{T}_1$ and $\|\mathcal{T}_1^*\|_{\mathcal{K}^*} = \|\mathcal{T}_1\|_{\mathcal{K}} \leqslant 1/2$. Using estimates (40) and (47), one finds that $\|\mathbf{g}'\|_{\mathcal{K}^*}, \|\mathbf{g}''\|_{\mathcal{K}^*} \leqslant 2$. It follows that:

$$\|\mathbf{f}' - \mathbf{f}''\|_{\mathcal{K}^*} \leqslant 8,$$

which yields the desired estimate (5) for operators $\widehat{u}_I$.

APPENDIX. GROUND STATES OF OPEN QUANTUM SYSTEMS

In this Appendix, we want to explain how a boundary condition can be explicitly defined for a state satisfying Definition 2. We adopt here a more abstract setting and also assume for simplicity all the Hilbert spaces to be finite dimensional. Let H be a Hamiltonian ($\equiv$ any self-adjoint operator) acting on a Hilbert space $\mathcal{H}$. As usual, a state ω on $\mathcal{B}(\mathcal{H})$ is said to be a ground state of H if it minimizes the expectation $\omega(H)$. It is a simple exercise to check that a state ω is a ground state if and only if

$$\omega(A^*[H, A]) \geqslant 0 \tag{A.1}$$

for any $A \in \mathcal{B}(\mathcal{H})$.

Let us now consider an open system with the Hilbert space $\mathcal{H}_i \otimes \mathcal{H}_b$, where $\mathcal{H}_i$ describes the "internal" part and $\mathcal{H}_b$ the "boundary" part. Suppose that its evolution is governed by a Hamiltonian H_1 (acting on $\mathcal{H}_i \otimes \mathcal{H}_b$), but the boundary can also interact with some "external" degrees of freedom forming a Hilbert space $\mathcal{H}_e$, via a Hamiltonian H_2 (acting on $\mathcal{H}_b \otimes \mathcal{H}_e$), so that

$$H = H_1 + H_2$$

is the full Hamiltonian, acting on $\mathcal{H}_i \otimes \mathcal{H}_b \otimes \mathcal{H}_e$. Suppose now that ω is a ground state of H. Let ω_1 be the restriction of ω to $\mathcal{B}(\mathcal{H}_i \otimes \mathcal{H}_b)$. Note that $[H, A] = [H_1, A]$ for $A \in \mathcal{B}(\mathcal{H}_i)$, so using (A.1) we find

$$\omega_1(A^*[H_1, A]) \geqslant 0 \text{ for } A \in \mathcal{B}(\mathcal{H}_i). \tag{A.2}$$

One can ask now if the converse holds: given a Hilbert space $\mathcal{H}_i \otimes \mathcal{H}_b$, a Hamiltonian H_1 on it and a state ω_1 obeying can one find a Hilbert space H_e and a Hamiltonian H_2 on $\mathcal{H}_b \otimes \mathcal{H}_e$ such that ω_1 is a restriction of a ground state of $H_1 + H_2$? It turns out to be true if "infinitely strong" interactions are allowed (we need to compactify the space of interactions, because the set of ground states is compact). We say that H_2 is a *generalized* Hamiltonian on $\mathcal{H}_b \otimes \mathcal{H}_e$ if H_2 is actually a self-adjoint operator on a subspace $\mathcal{G} \subset \mathcal{H}_b \otimes \mathcal{H}_e, \mathcal{G} \neq \{0\}$ (so that formally $(H_2 v, v) = +\infty$ for $v \in \mathcal{H}_b \otimes \mathcal{H}_e \setminus \mathcal{G}$). If H_1 is a usual Hamiltonian on $\mathcal{H}_i \otimes \mathcal{H}_b$, then the sum of H_1 and H_2 is a self-adjoint operator on $\mathcal{H}_i \otimes \mathcal{G}$, defined as a quadratic form

$$(Hv, v) = (H_1 v, v) + (H_2 v, v), \quad v \in \mathcal{H}_i \otimes \mathcal{G}. \tag{A.3}$$

Any state on $\mathcal{B}(\mathcal{H}_i \otimes \mathcal{G})$ (and in particular a ground state of H) can be considered as a state on $\mathcal{B}(\mathcal{H}_i \otimes \mathcal{H}_b \otimes \mathcal{H}_e)$ and hence restricted to $\mathcal{B}(\mathcal{H}_i \otimes \mathcal{H}_b)$. Now we can state

Proposition. Let H_1 be a Hamiltonian on $\mathcal{H}_i \otimes \mathcal{H}_b$ and ω_1 a state on $\mathcal{B}(\mathcal{H}_i \otimes \mathcal{H}_b)$. Then the two conditions are equivalent

(1) $\omega_1(A^*[H_1, A]) \geqslant 0$ for all $A \in \mathcal{B}(\mathcal{H}_i)$,

(2) there exists a Hilbert space $\mathcal{H}_e$ and a generalized Hamiltonian H_2 on $\mathcal{H}_b \otimes \mathcal{H}_e$ such that ω_1 is a restriction of a ground-state ω of H, defined as in (A.3), to $\mathcal{B}(\mathcal{H}_i \otimes \mathcal{H}_b)$.

Proof. We prove (1)$\Rightarrow$(2) (the converse implication is straightforward). We define ω simply as a purification of ω_1: choose some $\mathcal{H}_e$ and a unit vector $w \in \mathcal{H}_i \otimes \mathcal{H}_b \otimes \mathcal{H}_e$ such that $(Aw, w) = \omega_1(A)$ for all

$A \in \mathcal{B}(\mathcal{H}_i \otimes \mathcal{H}_b)$, then set $\omega(\cdot) := (\cdot w, w)$. Next we take a biorthogonal decomposition of w

$$w = \sum_k u_k \otimes v_k,$$

where $\{u_k\}$ is an orthonormal family in $\mathcal{H}_i$ and $\{v_k\}$ an orthogonal family in $\mathcal{H}_b \otimes \mathcal{H}_e$ with all $\|v_k\| > 0$. Let subspaces $\mathcal{F}$ and $\mathcal{G}$ be spanned by $\{u_k\}$ and $\{v_k\}$, respectively. Below we will define the operator H_2 on the subspace $\mathcal{G}$. Let $H_1 w = w' = w_1' + w_2'$, where w_1' and w_2' are projections of w' on $\mathcal{H}_i \otimes \mathcal{G}$ and $\mathcal{H}_i \otimes (\mathcal{H}_b \otimes \mathcal{H}_e \ominus \mathcal{G})$, respectively. Note that condition (1) implies

$$\omega_1([H_1, A]) = 0 \qquad \text{for } A \in \mathcal{B}(\mathcal{H}_i)$$

(stationarity of ω_1). Indeed, this follows by substituting $A + t$ for A in condition 1) and equating in the resulting, linear in t expression the t coefficient to 0. Now if $\operatorname{Ran} A \subset \mathcal{H}_i \ominus \mathcal{F}$ then $\mathcal{F} \subset \ker A^*$ and hence

$$0 = \omega_1([H_1, A]) = (Aw, H_1 w) - (H_1 w, A^* w) = (Aw, w') = (Aw, w_1').$$

It follows that $w_1' \in \mathcal{F} \otimes \mathcal{G}$ and therefore there exists a (unique) linear operator H_2 on $\mathcal{G}$ such that $w_1' = -H_2 w$. Let us show that H_2 is self-adjoint. Let A_{kl} be a matrix unit in the $\{u_k\}$ basis

$$A_{kl} u_k = u_l, \;\; A_{kl}|_{\mathcal{H}_i \ominus u_k} = 0$$

so that $A_{kl}^* = A_{lk}$. Using again the stationarity, we have

$$0 = ([H_1, A_{kl}]w, w) = (A_{kl} w, w_1') - (w_1', A_{lk} w) = -(v_k, H_2 v_l) + (H_2 v_k, v_l),$$

which proves the self-adjointness. Now it remains to check that if H is defined by (A.3) then w is its ground-state vector. Since $[H_2, A] = 0$ for $A \in \mathcal{B}(\mathcal{H}_i)$, by condition (1) for such A

$$0 \leqslant ([H_1 + H_2, A]w, Aw) = ((H_1 + H_2)Aw, Aw),$$

because $(H_1 + H_2)w = w_2' \in \mathcal{H}_i \otimes (\mathcal{H}_b \otimes \mathcal{H}_e \ominus \mathcal{G})$, whereas $\{Aw | A \in \mathcal{B}(\mathcal{H}_i)\} = \mathcal{H}_i \otimes \mathcal{G}$. It follows that if H is defined as in (A.3), then $Hw = 0$ and $(Hv, v) \geqslant 0$ for all $v \in \mathcal{H}_i \otimes \mathcal{G}$, which proves that ω is a ground state of H. ▮

The above Proposition makes the meaning of Definition 2 more clear. The inner spins from Λ^o interact only with other spins in Λ as prescribed by the formal Hamiltonian, while the boundary spins from $\Lambda \setminus \Lambda^o$ are allowed to interact with some external degrees of freedom in an arbitrary manner. Definition 2 then describes the restriction of a ground state of the whole system to $\mathcal{B}(\mathcal{H}_\Lambda)$.

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